

About the reducibility of the variety of complex Leibniz algebras

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Abstract

In this paper, using the notions of perturbation and contraction of Lie and Leibniz algebras, we show that the algebraic varieties of Leibniz and nilpotent Leibniz algebras of dimension greater than 2 are reducible.

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1 Definition and preliminary properties

The aim of this work is to prove the reducibility of the Leibniz and nilpotent Leibniz algebraic varieties, that we will denote Leib^n and LeibN^n respectively. First we will classify the 3-dimensional nilpotent Leibniz algebras over the complex field. Then, using the internal set theory of Nelson [8], we will introduce what a perturbation of a Leibniz algebra is. Such notion allows us to determine the open components of the variety LeibN^3 , it will turn out to have two of them. The other algebras are obtained as the limit by contraction of the rigid algebra or the family of rigid algebras defining the open components. Moreover, we characterize the rigid Lie algebras over the variety L^n of Lie algebras that are rigid over Leib^n .

Definition 1 A *Leibniz algebra law* μ over \mathbb{C} is a bilinear map $\mu : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ satisfying

$$\mu(x, \mu(y, z)) = \mu(\mu(x, y), z) - \mu(\mu(x, z), y). \quad (1)$$

We call **Leibniz algebra** to any pair (\mathbb{C}^n, μ) where μ is a Leibniz algebra law.

The previous equation is known as the Leibniz identity. From now on, the law will be identified with its algebra and the non written products will be supposed to be zero.

Notice that if μ is anticommutative $\mu(x, y) = -\mu(y, x)$, the Leibniz identity is equivalent to the Jacobi one

$$\mu(x, \mu(y, z)) + \mu(y, \mu(z, x)) + \mu(z, \mu(x, y)) = 0, \quad (2)$$

as (1) is obtained by placing the element x on the first place at every element of the Jacobi identity.

Let $\mathfrak{l} = (\mathbb{C}^n, \mu)$ be a Leibniz algebra, we define the **right-decreasing central sequence** as

$$\mathcal{C}^1(\mathfrak{l}) = \mathfrak{l} \quad \mathcal{C}^2(\mathfrak{l}) = \mu(\mathfrak{l}, \mathfrak{l}) \quad \dots \quad \mathcal{C}^{k+1}(\mathfrak{l}) = \mu(\mathcal{C}^k(\mathfrak{l}), \mathfrak{l}) \quad \dots$$

Definition 2 A Leibniz algebra \mathfrak{l} is **nilpotent** if there exists some $k \in \mathbb{N}$ such that $\mathcal{C}^k(\mathfrak{l}) = \{0\}$.

For a given nilpotent Leibniz algebra $\mathfrak{l} = (\mathbb{C}^n, \mu)$, we define for every $x \in \mathbb{C}^n$ the endomorphism $R_x : \mathbb{C}^n \rightarrow \mathbb{C}^n$ as

$$R_x(y) = \mu(y, x), \quad \forall y \in \mathbb{C}^n.$$

It is easy to check that R_x is a nilpotent endomorphism, then for any $x \in \mathfrak{l} \setminus \mathcal{C}^2(\mathfrak{l})$, we write $s_\mu(x) = (s_1(x), \dots, s_k(x))$ the decreasing sequence $s_1 \geq s_2 \geq \dots \geq s_k$ of dimensions of the Jordan blocks of the nilpotent operator R_x . We may now order lexicographically $s_\mu(x)$ for all $x \in \mathfrak{l} \setminus \mathcal{C}^2(\mathfrak{l})$ and denote $s(\mu)$ its maximum which is, up to isomorphism, an invariant of the isomorphism class of the algebra \mathfrak{l} . We call it **characteristic sequence** of \mathfrak{l} . If $x \in \mathfrak{l} \setminus \mathcal{C}^2(\mathfrak{l})$ satisfies $s_\mu(x) = s(\mu)$, we say that x is a **characteristic vector** of \mathfrak{l} .

We will denote Leib^n the set of all Leibniz algebras over \mathbb{C}^n and LeibN^n the set of nilpotent Leibniz algebras over \mathbb{C}^n .

Notice that we can identify any Leibniz algebra μ with its structure constants over a fixed base. Given $\{e_1, \dots, e_n\}$ a base of \mathbb{C}^n , from the identity (1) we have that the coordinates defined by $\mu(e_i, e_j) = a_{ij}^k e_k$ are the solution of

$$a_{jk}^l a_{il}^m - a_{ij}^l a_{lk}^m + a_{ik}^l a_{lj}^m = 0, \quad 1 \leq i, j, k, m \leq n \quad (3)$$

As the nilpotent conditions are also polynomials, Leib^n and LeibN^n can be endowed with an algebraic structure over \mathbb{C}^{n^3} .

2 Classification of the nilpotent Leibniz algebra of dimension 3

Let $\mathfrak{l} = (\mathbb{C}^3, \mu)$ be a nilpotent Leibniz algebra. According to the previous section, the possible characteristic sequences of \mathfrak{l} are $\mathcal{C}(\mathfrak{l}) \in \{(3), (2, 1), (1, 1, 1)\}$.

1. If $s(\mathfrak{l}) = (3)$, there exists a characteristic vector e_1 and a base $\{e_1, e_2, e_3\}$ such that

$$\begin{aligned} \mu(e_1, e_1) &= e_2, \\ \mu(e_2, e_1) &= e_3. \end{aligned}$$

As $\mu(x, e_2) = \mu(x, \mu(e_1, e_1)) = \mu(\mu(x, e_1), e_1) - \mu(\mu(x, e_1), e_1) = 0$, we have that $R_{e_2} = 0$. The Leibniz identity for (e_1, e_2, e_1) , (e_2, e_2, e_1) and (e_3, e_2, e_1) shows that $\mu(e_1, e_3) = \mu(e_2, e_3) = \mu(e_3, e_3) = 0$. In this case thus, there only exists (up to isomorphism) one nilpotent Leibniz algebra μ_1 of maximal characteristic sequence, which is given by

$$\begin{aligned} \mu_1(e_1, e_1) &= e_2, \\ \mu_1(e_2, e_1) &= e_3. \end{aligned}$$

2. If $s(\mathfrak{l}) = (2, 1)$, we have two possibilities

- (a) There exists a characteristic vector e_1 such that $\mu(e_1, e_1) \neq 0$.
- (b) For every characteristic vector x , we have $\mu(x, x) = 0$.

For the (a) case, we can find a base $\{e_1, e_2, e_3\}$ such that

$$\begin{aligned} \mu(e_1, e_1) &= e_2, \\ \mu(e_2, e_1) &= 0, \\ \mu(e_3, e_1) &= 0. \end{aligned}$$

The Leibniz identity for (x, e_1, e_1) leads again to $\mu(x, e_2) = 0$, whereas using it for (e_2, e_2, e_3) and the nilpotency of R_{e_3} lead to $\mu(e_2, x) = 0$. Finally the Leibniz identity for (e_1, e_1, e_3) and (e_3, e_3, e_3) implies that

$$\begin{aligned}\mu(e_1, e_3) &= ae_2, \\ \mu(e_3, e_3) &= be_2.\end{aligned}$$

If we consider a change of base $\{x_1, x_2, x_3\}$ such that $\mu(x_2, x_1) = 0$, $\mu(x_3, x_1) = 0$ and $\mu(x_2, x_3) = 0$, we remain in this family of nilpotent Leibniz algebras, and the nullity or non nullity of a and b are preserved under this change of basis. Then, if $a \neq 0$ considering $x_1 = e_1$, $x_2 = e_2$ and $x_3 = \frac{1}{a}e_3$, leads to the family of non isomorphic Leibniz algebras $\mu_{2,b}$ given by

$$\begin{aligned}\mu_{2,b}(e_1, e_1) &= e_2, \\ \mu_{2,b}(e_3, e_3) &= be_2, \\ \mu_{2,b}(e_1, e_3) &= e_2.\end{aligned}$$

If $a = 0$ but $b \neq 0$, we can analogously take $b = 1$ leading to the sole algebra μ_3 given by

$$\begin{aligned}\mu_3(e_1, e_1) &= e_2, \\ \mu_3(e_3, e_3) &= e_2.\end{aligned}$$

Finally if $a = b = 0$ we obtain the algebra μ_4 given by $\mu_4(e_1, e_1) = e_2$

For the (b) case, there exists a basis $\{e_1, e_2, e_3\}$ such that $\mu(e_2, e_1) = e_3$. The nilpotency of R_{e_3} , R_{e_2} and the fact that there is no characteristic vector x such that $\mu(x, x) \neq 0$, imply that the Leibniz algebra μ is in fact a Lie algebra isomorphic to the Heisenberg algebra of dimension 3 i.e. μ is isomorphic to μ_5 given by $\mu_5(e_1, e_2) = -\mu_5(e_2, e_1) = -e_3$.

3. If $s(\mathfrak{l}) = (1, 1, 1)$, it turns out that μ is the abelian algebra $\mu_6 = 0$.

The previous analysis shows the following result

Theorem 1 *Every nilpotent complex Leibniz algebra is isomorphic to one of the algebras μ_i with $i = 1, 3, 4, 5, 6$ or to $\mu_{2,b}$ with $b \in \mathbb{C}$.*

3 Contractions and perturbations of the Leibniz algebras

In this section \mathcal{L}^n will denote the variety of Lie algebras L^n , or one of the varieties Leib^n or LeibN^n . If $\mu_0 \in \mathcal{L}^n$, we denote $\mathcal{O}(\mu_0)$ the orbit of μ_0 under the action of the general linear group $GL(n, \mathbb{C})$ over \mathcal{L}^n :

$$\begin{aligned}GL(n, \mathbb{C}) \times \mathcal{L}^n &\longrightarrow \mathcal{L}^n \\ (f, \mu_0) &\longmapsto f^{-1} \circ \mu_0 \circ (f \times f)\end{aligned}$$

where $f^{-1} \circ \mu_0 \circ (f \times f)(x, y) = f^{-1}(\mu_0(f(x), f(y)))$.

Let C be an irreducible component of \mathcal{L}^n containing μ_0 , then $\mathcal{O}(\mu_0) \subseteq C$. We can endow naturally the variety \mathcal{L}^n with two non equivalent topologies: the metric topology induced by the inclusion of \mathcal{L}^n in \mathbb{C}^{n^3} , and the Zariski topology. Notice that the latter is contained in the former. As C is closed in the Zariski topology, the adherence $\overline{\mathcal{O}(\mu_0)}^Z$ of the orbit μ_0 is also contained in C .

In analogy with the Lie algebras we can formally define the notion of limit over the variety \mathcal{L}^n as follows: let $f_t \in GL(n, \mathbb{C})$ be a family of non-singular endomorphism depending on a continuous parameter t , and consider some $\mu \in \mathcal{L}^n$. If for every pair $x, y \in \mathbb{C}^n$ the limit

$$\mu'(x, y) := \lim_{t \rightarrow 0} \mu_t(x, y) := \lim_{t \rightarrow 0} f_t^{-1} \circ \mu(f_t(x), f_t(y)) \quad (4)$$

exists, then μ' is an algebra law of \mathcal{L}^n . We call this new law the **contraction** of μ by $\{f_t\}$. Using the action of $GL(n, \mathbb{C})$ over the variety \mathcal{L}^n , it is easy to see that a contraction of μ corresponds to a point of the closure of the orbit $\mathcal{O}(\mu)$.

It is important to notice that any non trivial contraction $\mu \rightarrow \mu'$ satisfies

$$\begin{aligned} \dim \mathcal{O}(\mu) &> \dim \mathcal{O}(\mu'), \\ \dim Z_R(\mu) &\leq \dim Z_R(\mu') & \text{where } Z_R(\mu) &= \{x \in \mathbb{C}^n : \mu(y, x) = 0, \forall y \in \mathbb{C}^n\}, \\ s(\mu) &\geq s(\mu') & \text{in the nilpotent case.} \end{aligned}$$

Therefore every component C containing μ_0 contains also any of its contractions.

Definition 3 Assuming the non standard analysis (I.S.T.) of Nelson [8], let μ_0 be a standard law of \mathcal{L}^n . A perturbation μ of μ_0 over \mathcal{L}^n is another law in \mathcal{L}^n satisfying the condition $\mu(x, y) \sim \mu_0(x, y)$ for every standard x, y over \mathbb{C}^n , where $a \sim b$ means that the vector $a - b$ is infinitesimally small.

In particular if $\mu' = \lim_{t \rightarrow 0} \mu_t$ is a contraction of μ , for every t_0 infinitesimally small, the law μ_{t_0} is isomorphic to μ and is in fact a perturbation of μ' . Such remark encodes perfectly the link between the notions of perturbation and contraction.

Consequence. The invariants of the nilpotent laws characterizing the irreducibles components are the stable invariants under perturbation. In particular if $\tilde{\mu}$ is a perturbation of μ , then

$$\begin{aligned} \dim \mathcal{O}(\tilde{\mu}) &> \dim \mathcal{O}(\mu), \\ \dim Z_R(\tilde{\mu}) &\leq \dim Z_R(\mu), \\ s(\tilde{\mu}) &\geq s(\mu) & \text{in the nilpotent case.} \end{aligned}$$

Definition 4 A standard law $\mu \in \mathcal{L}^n$ is **rigid** over \mathcal{L}^n if any perturbation of μ is isomorphic to it.

This definition translates to the non-standard language the classic notion of rigidity. In fact, if any perturbation of μ is isomorphic to μ , its halo (i.e. the class of laws μ' such that $\mu' \sim \mu$) is contained on the orbit $\mathcal{O}(\mu)$. This implies that the orbit is open and, by the transfer principle, we obtain the equivalence. In particular we obtain that the rigid algebras cannot be obtained by contraction and that the rigidity of $\mu \in \mathcal{L}^n$ over \mathcal{L}^n implies that $\overline{\mathcal{O}(\mu)}^Z$ is an irreducible component of the variety \mathcal{L}^n .

4 The variety LeibN³

In this section, using the notions of the previous paragraph, we determine the irreducibles components of the variety LeibN³.

1. The law μ_1 (sec. 2) is rigid over LeibN³. Clear as it is the only nilpotent Leibniz algebra with a maximal characteristic sequence.

2. $\mu_{2,b}$ ($b \neq 0$), μ_3 and μ_5 are not contractions of μ_1 . The dimension of the center cannot decrease with a contraction, however $\dim(Z_R(\mu_1)) = 2$, $\dim(Z_R(\mu_{2,b})) = 1$ (in the $b \neq 0$ case), $\dim(Z_R(\mu_3)) = 1$ and $\dim(Z_R(\mu_5)) = 1$.
3. The only contractions of μ_1 are isomorphic to $\mu_{2,0}$, μ_4 and μ_6 . It is enough to consider the following family of automorphisms of \mathbb{C}^3

$$\left\{ \begin{array}{l} f_t(e_1) = te_1 \\ f_t(e_2) = t^2e_2 \\ f_t(e_3) = e_3 + te_1 \end{array} \right. \quad \left\{ \begin{array}{l} g_t(e_1) = te_1 \\ g_t(e_2) = t^2e_2 \\ g_t(e_3) = e_3, \end{array} \right. \quad \left\{ \begin{array}{l} h_t(e_1) = te_1 \\ h_t(e_2) = te_2 \\ h_t(e_3) = te_3, \end{array} \right.$$

to obtain the contractions of μ_1 into $\mu_{2,0}$, μ_4 and μ_6 respectively.

4. If $b \neq 0$ and $\tilde{\mu}$ is a perturbation of $\mu_{2,b}$, there exists some $b' \in \mathbb{C}$ such that $\tilde{\mu}$ is isomorphic to $\mu_{2,b'}$. This means that the family $\{\mu_{2,b}\}_{b \neq 0}$ is rigid.

In fact, on one hand we have that $\tilde{\mu} \notin \mathcal{O}(\mu_1)$ and thus $s(\tilde{\mu}) = (2, 1)$. On the other hand, by the transfer property [8] we can assume that b , $\mu_{2,b}$ and $\{e_1, e_2, e_3\}$ are standard and therefore

$$\begin{aligned} \tilde{\mu}(e_1, e_1) &\sim \mu_{2,b}(e_1, e_1), \\ \tilde{\mu}(e_1, e_3) &\sim \mu_{2,b}(e_1, e_3), \\ \tilde{\mu}(e_3, e_3) &\sim \mu_{2,b}(e_3, e_3), \end{aligned}$$

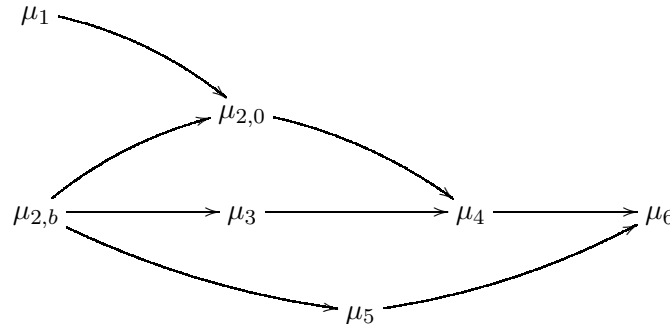
and the result follows.

5. The algebras $\mu_{2,0}$, μ_3 , μ_4 , μ_5 and μ_6 can be perturbed over the laws of the family $\{\mu_{2,b}\}_{b \neq 0}$. In order to obtain perturbed algebras isomorphic to the one of the family $\{\mu_{2,b}\}_{b \neq 0}$, it is enough to consider the bilinear maps defined by

$$\begin{aligned} \varphi_2(e_3, e_3) &= e_2, & \varphi_3(e_1, e_3) &= e_2, \\ \varphi_4(e_3, e_3) &= \varphi_3(e_1, e_3) = e_2, & \varphi_5(e_1, e_1) &= e_1, \end{aligned}$$

and the laws of LeibN³ given by $\mu_{2,0} + \varepsilon\varphi_2$, $\mu_i + \varepsilon\varphi_i$ for $i = 3, 4, 5$, where $\varepsilon \sim 0$ is non zero.

Analogously, we can show that the only contraction of μ_3 and $\mu_{2,0}$ are isomorphic to μ_4 and μ_6 , and that the only contraction of μ_4 and μ_5 is μ_6 . We can summarize all these results in the following diagram, where the arrows represent contractions and hence the rigid elements are those for which no arrow finishes at them



After this study, we can classify the components of the variety as follows

Theorem 2 *The variety LeibN^3 is the union of the two irreducibles components $\overline{\mathcal{O}(\mu_1)}^Z$ and $\bigcup_{b \in \mathbb{C}} \overline{\mathcal{O}(\mu_{2,b})}^Z$.*

Remark 1 *In reference [1], the authors claim that the law λ_5 of LeibN^3 defined (over the basis $\{x_1, x_2, x_3\}$) by*

$$\lambda_5(x_2, x_2) = \lambda_5(x_3, x_2) = \lambda_5(x_2, x_3) = x_1,$$

is rigid. Notice however that λ_5 is isomorphic to μ_3 via the change of basis $e_1 = x_2, e_2 = x_1$ and $e_3 = -ix_2 + ix_3$. As μ_3 can be perturbed into the family $\{\mu_{2,b}\}$, such claim is not correct.

5 The reducibility of the varieties LeibN^n and Leib^n

Let $\mu_0 \in \text{LeibN}^n$ be a law with characteristic sequence $s(\mu_0) = (n)$. In that case there exists a basis $\{e_1, \dots, e_n\}$ of \mathbb{C}^n such that $\mu_0(e_i, e_1) = e_{i+1}$ for $i = 1, 2, \dots, n-1$. Once again, applying the Leibniz identity to (x, e_1, e_1) we obtain that $R_{e_2} = 0$. In fact every $R_{e_k} = 0$ for $k \geq 2$, as by induction $\mu_0(x, e_{k+1}) = \mu_0(x, \mu(e_1, e_k)) = \mu_0(\mu_0(x, e_1), e_k) - \mu_0(\mu_0(x, e_k), e_1) = 0$.

Proposition 1 *Any nilpotent Leibniz algebra μ of dimension n and characteristic sequence $s(\mu) = (n)$ is isomorphic to μ_0 , where $\mu_0(e_i, e_1) = e_{i+1}$ for $i = 1, \dots, n-1$.*

Remark 2 *The law μ satisfying $\dim(\mathcal{C}^i(\mu)) - \dim(\mathcal{C}^{i+1}(\mu)) = 1$ for every $i = 1, \dots, n$ is called **null-filiform** in reference [4].*

As μ_0 is the nilpotent Leibniz algebra with a maximal characteristic sequence, it has to be rigid and $\overline{\mathcal{O}(\mu_0)}^Z$ is an irreducible component of the variety LeibN^n . On the other hand if μ is a non abelian Lie algebra, $\dim(Z(\mu)) \leq n-2$ and $\dim(Z_R(\mu_0)) = n-1$ implying that μ cannot be a contraction of μ_0 . From this considerations we have the following theorem

Theorem 3 *The variety LeibN^n for $n \geq 3$ is reducible.*

Remark 3 *LeibN^2 is irreducible and the only irreducible component is $\overline{\mathcal{O}(\mu)}^Z$, where μ is the law defined over the basis $\{e_1, e_2\}$ by $\mu(e_1, e_1) = e_2$.*

Let $\mathfrak{l} = (\mathbb{C}^n, \mu)$ be a Leibniz algebra. It is clear that $Z_R(\mu)$ is an ideal of \mathfrak{l} that contains the elements of the form $\mu(x, y) + \mu(y, x)$, $\mu(x, x)$ and $\mu(\mu(x, y), \mu(y, x))$ with $x, y \in \mathbb{C}^n$. Thus $\mathfrak{l}/Z_R(\mu)$ is a Lie algebra, which shows the following claim

Every Leibniz algebra which is not a Lie algebra verifies $Z_R(\mu) \neq 0$.

Theorem 4 *A L^n -rigid Lie algebra without center is also rigid over Leib^n .*

Proof. Let μ be a rigid Lie algebra without center. Let $\tilde{\mu}$ be a perturbation of μ in LeibN^n . As $Z(\mu) = 0$ then $Z_R(\tilde{\mu}) = 0$ and $\tilde{\mu} \in L^n$. By the rigidity of μ over L^n , $\tilde{\mu}$ is isomorphic to μ . ■

Theorem 5 *A Lie algebra with non null center cannot be rigid over Leib^n .*

Proof.

Let μ be a Lie algebra with non null center. We may assume n and μ_0 standard. Let x be a generator of the Lie algebra, y a non zero vector of the center and φ the bilinear algebra such that its only non vanishing product is $\varphi(x, x) = y$. Thus the perturbation $\tilde{\mu}$ of μ given by $\tilde{\mu} = \mu + \varepsilon\varphi$ where $\varepsilon \sim 0$ is non zero, is a Leibniz algebra that is not a Lie algebra, then $\tilde{\mu}$ cannot be isomorphic to μ . ■

Corollary 1 *The variety Leib^n is reducible for $n \geq 2$. In fact,*

- Leib^6 has at least 5 irreducible components,
- Leib^7 has at least 8 irreducible components,
- Leib^8 has at least 33 irreducible components.
- Leib^9 has at least 41 irreducible components.

For $n \geq 81$, the number of irreducible components of Leib^n is lower from below by $\Gamma(\sqrt{n})$, where Γ is Euler gamma function (see [5] and [6]).

Remark 4 *The variety Leib^2 is the union of the two irreducibles components $\overline{\mathcal{O}(\varphi_1)}^Z$ and $\overline{\mathcal{O}(\varphi_2)}^Z$, where the laws are defined, in the basis $\{e_1, e_2\}$ of \mathbb{C}^2 , by $\varphi_1(e_1, e_2) = -\varphi_1(e_2, e_1) = e_2$ (Lie algebra) and $\varphi(e_2, e_1) = e_2$.*

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